First semestral examination 2009
B.Math.Hons.First year
Algebra — B.Sury
Each question carries 10 marks

Q 1.
Let H be a subgroup of a finite group G.
(i) Show that any p-Sylow subgroup of H can be obtained as the intersection of H with a p-Sylow subgroup of G.
(ii) If H is normal in G, show that for each p-Sylow subgroup P of G, $P \cap H$ is a p-Sylow subgroup of H.

Q 2.
(i) Prove that the number of group homomorphisms from $\mathbf{Z}_m$ to $\mathbf{Z}_n$ is (m, n).
(ii) Use the class equation to prove that there is no finite group of order > 2 with exactly two conjugacy classes.

Q 3.
Determine the number of elements of order 7 in a simple group of order 168.

Q 4.
Let G be a group of odd order in which $(xy)^{1024} = x^{1024}y^{1024}$ for all $x, y \in G$. Prove that every element of G is a 1024-th power.

Q 5.
Let G be a group and let $g \in G$ have order 17. If $x \in N_G(< g >)$ has odd order, prove that $xg = gx$.

Q 6.
Show that the polynomial $X^2 + 1$ is irreducible in $\mathbf{Z}_{11}[X]$. Hence deduce that there is a field with 121 elements.

Q 7.
Consider the ring $R := \{\frac{a}{b} \in \mathbf{Q} : 3 \not| b\}$. Show that $M := \{3x : x \in R\}$ is the unique maximal ideal of R.
Hint : Show that elements outside M are units.
. P.T.O.

Q 8.
Show that $\mathbf{Z}[X]$ is not a principal ideal domain.

Q 9.
Let R be an integral domain where it is unknown if it contains a unity.
(i) If there is $0 \neq e \in R$ such that $e^2 = e$, prove that e is a unity in R.
(ii) If $na = 0$ for some $a \neq 0$ and some natural number n, prove that $nb = 0$ for all $b \in R$.

Q 10.
Give one-line reasons (proofs not needed) for the following :
(i) A group of order 1024 cannot be simple.
(ii) There is no nontrivial group G for which $G/Z(G)$ is of order 5.
(iii) Given that S_5 has a subgroup isomorphic to the dihedral group D_8 of order 8, the quaternion group Q_8 cannot be a 2-Sylow subgroup of S_5.
(iv) In the ring $R = \mathbf{Z}_{22}$, the subring $\{2x : x \in R\}$ has a unity.
(v) If $I_1 \subset I_2 \subset \cdots\cdots \subset R$ is an increasing sequence of proper ideals in a commutative ring R, then $\bigcup_{n \geq 1} I_n$ is a proper ideal.